A Snippet of Morse Theory

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1 Morse Functions

1.1 Notation

We consider functions $f : \mathbb{R}^2 \to \mathbb{R}^+$ as models for images. Define the **gradient** as

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x) \right)^T \text{ where } x = (x_1, x_2),$$

and define the **Hessian** as

$$\nabla^2 f(x) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) \end{pmatrix}.$$

Define a **level set** of a function at level $a \in \mathbb{R}^+$ as

$$f^{-1}(a) := \{x \in \mathbb{R}^2 : f(x) = a\}.$$

1.2 Basics of Morse Functions

Definition 1 (Critical Point)**.**
- *For a C^1 function, a* **critical point** $p \in \mathbb{R}^2$ *is a point such that* $\nabla f(p) = 0$.
- *A critical point p is a* **local minimum point** *if $\exists\, \delta > 0$ such that $f(x) \geq f(p)$ for x such that $|x - p| < \delta$.*
- *A critical point p is a* **local maximum point** *if $\exists\, \delta > 0$ such that $f(x) \leq f(p)$ for x such that $|x - p| < \delta$.*
- *A critical point p is a* **saddle point** *if it is neither a local minimum or local maximum.*

Definition 2 (Morse Function)**.** *A* **Morse function** *f is a C^2 function such that all critical points are non-degenerate, i.e., if p is a critical point, then* $\det \nabla^2 f(p) \neq 0$.

Remark 1.
- *By Taylor's Theorem, we see that Morse functions are well approximated by quadratic forms around critical points:*

$$\begin{aligned} f(x) &= f(p) + \nabla f(p) \cdot (x-p) + (x-p)^T \nabla^2 f(p)(x-p) + o(|x-p|^2) \\ &= (x-p)^T \nabla^2 f(p)(x-p) + o(|x-p|^2) \end{aligned}$$

provided that $f(p) = 0$ (if not, set f to $f - f(p)$).

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- *In particular, this means that Morse functions have* **isolated** *critical points.*

The previous Remark, leads to the following observation:

Theorem 1 (Morse Lemma)**.** *If f is a Morse function, then for a critical point p of f, there is a neighborhood $U \ni p$ and a chart (coordinate change) $\psi : \tilde{U} \subset \mathbb{R}^2 \rightarrow U \subset \mathbb{R}^2$ so that*

$$f(\hat{x}) = f(p_i) + \begin{cases} -(\hat{x}_1^2 + \hat{x}_2^2) & \text{if } p \text{ is a maximum point} \\ \hat{x}_1^2 + \hat{x}_2^2 & \text{if } p \text{ is a minimum point} \\ \hat{x}_1^2 - \hat{x}_2^2 & \text{if } p \text{ is a saddle point} \end{cases}$$

where $(\hat{x}_1, \hat{x}_2) = \psi_i(x_1, x_2)$ and $(x_1, x_2) \in \mathbb{R}^2$ are the natural arguments of f.

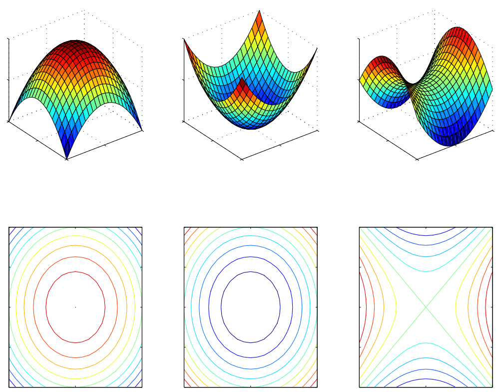


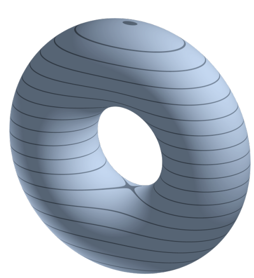
Figure 1: The Morse Lemma states that in a neighborhood of a critical point of a Morse function, the level sets are topologically equivalent to one of the three forms (left to right: maximum, minimum, and saddle critical point neighborhoods).

Remark 2. *Morse originally used the previous Theorem as the definition of Morse functions. Defining in that way does not require that a Morse function be differentiable.*

1.3 Examples of Morse/non-Morse Functions

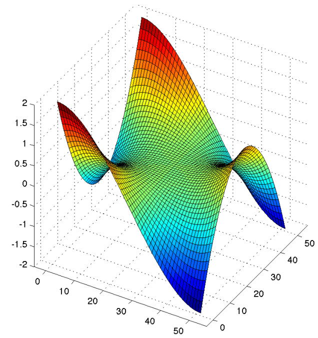
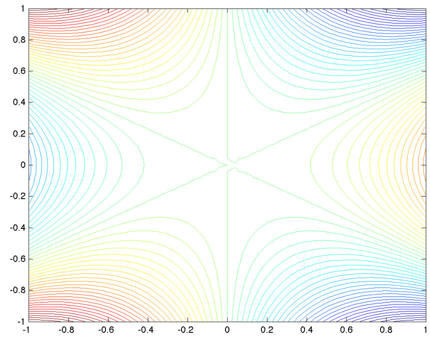
Examples of Morse/Non-Morse functions are the following:

- Obviously, $f(x_1, x_2) = x_1^2 + x_2^2$ and $f(x_1, x_2) = x_1^2 - x_2^2$ are Morse Functions.
- The *height function* $f : S \rightarrow \mathbb{R}^2$ of the embedding of a compact surface $S \subset \mathbb{R}^3$ without boundary is a Morse function. For example, the height function of the torus:



- $f(x_1, x_2) = x_1^4 + x_2^4$ is not a Morse function (degenerate critical point).

- A monkey saddle, i.e., $f(x_1, x_2) = x_1^3 - 3x_1x_2^2$:

is not a Morse function (degenerate; level sets of saddle must cross at an 'X').

- All non-smooth functions are not Morse functions, e.g., images that have edges!
- Functions that have co-dimension one critical sets (e.g., ridges and valleys) are not Morse functions. Such critical sets are commonplace in images!

1.4 Morse Functions are (Almost) all Functions

Morse functions seems to be a very restricted class of functions from the previous examples, however, a basic result from Morse Theory says that Morse functions are essentially *all* functions!

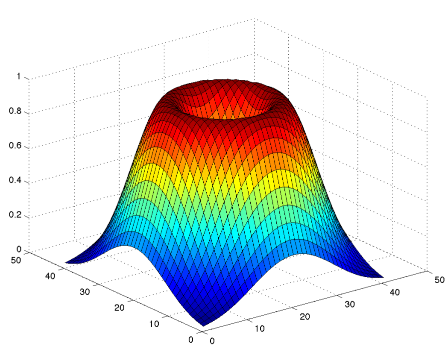
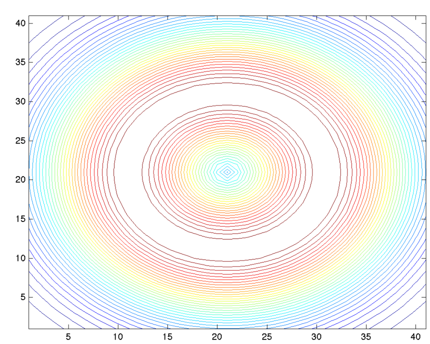
Definition 3 (Dense Subset of a Normed Space)**.** *Let $\|\cdot\|$ denote a norm on a topological space X. A set $S \subset X$ is* **dense** *if closure$(S) = X$, i.e., for every $x \in X$ and $\delta > 0$, there exists $s \in S$ such that $\|s - x\| < \delta$.*

Theorem 2 (Morse Functions are Dense)**.** *Let $\|\cdot\|$ denote the C^2 norm on the space of C^2 functions, i.e.,*

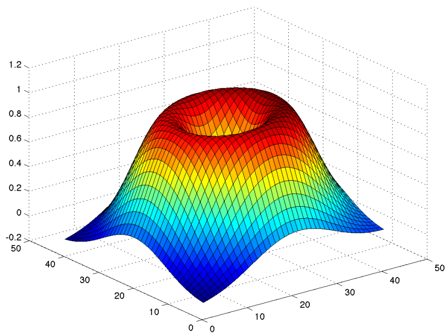
$$\|f\| = \sup_{x \in \mathbb{R}^2} |f(x)| + |\nabla f(x)| + |\nabla^2 f(x)|.$$

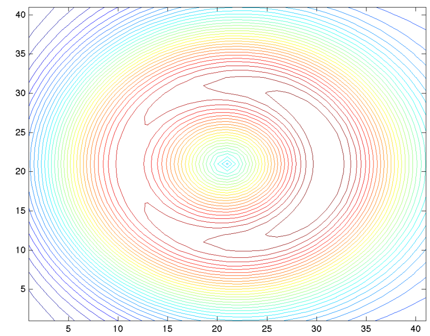
Then Morse functions form an open, dense subset of C^2 functions.

Remark 3.
- *Therefore, any smooth (C^2) function can be well approximated by a Morse function up to* arbitrary precision *(defined by $\|\cdot\|$).*
- *For example, ridges and valleys can be approximated with a Morse function, e.g., consider the following circular ridge that can be made Morse by a slight tilt. Let $f(x_1, x_2) = \exp\left(-(\sqrt{x_1^2 + x_2^2} - 1)^2\right)$ which is a ridge and non-Morse:*

Now consider the function $g(x_1, x_2) = f(x_1, x_2) + \varepsilon x_1$, which is arbitrarily close to f (in C^2 norm) and is a Morse function:





- *Note it is a basic fact that C^2 functions under the norm above are* dense *in all square integrable functions (L^2) functions. Therefore, even non-smooth functions can be approximated to arbitrary precision by Morse functions.*

2 Reeb Graph

2.1 Basic Definitions

Definition 4 (Equivalence Relation)**.** *Let X be a set. An* **equivalence relation** *on X denoted $\sim$ is a binary relation with the following properties: for all $x, y, z \in X$:*

- *(reflexivity) $x \sim x$*
- *(symmetry) if $x \sim y$ then $y \sim x$*
- *(transitivity) if $x \sim y$ and $y \sim z$, then $x \sim z$*

We denote by $[x]$ all elements of X related to x, e.g., $[x] = \{y \in X : x \sim y\}$.

Definition 5 (Topological Space)**.** *A* **topology** *denoted $\mathcal{T}$ on a set X is a collection of subsets of X (called* open sets*) such that the following properties hold:*

- $\emptyset, X \in \mathcal{T}$
- *for $U_\alpha \in \mathcal{T}$ where $\alpha \in \mathcal{J}$ is an index set (perhaps uncountable), we have $\bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$.*
- *for $U_i \in \mathcal{T}$ where $i \in \mathcal{I}$ is a finite index set, we have $\bigcap_{i \in \mathcal{I}} U_i \in \mathcal{T}$.*

Definition 6 (Quotient Space)**.** *Let X be a topological space. Let $\sim$ denote an equivalence relation on X. The* **quotient space** *of X under the equivalence relation $\sim$, denoted $X/\sim$ is the topological space whose elements are*

$$X/\sim := \{[x] : x \in X\},$$

and whose topology is induced from X. The **quotient map** *is the (continuous) function $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$.*

Definition 7 (Reeb Graph)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Define a equivalence relation $\sim$ on the space $Graph(f) := \{(x, f(x)) : x \in \mathbb{R}^2\}$ by*

$$(x, f(x)) \sim (y, f(y)) \text{ iff } f(x) = f(y) \text{ and there is a continuous path from } x \text{ to } y \text{ in } f^{-1}(f(x)).$$

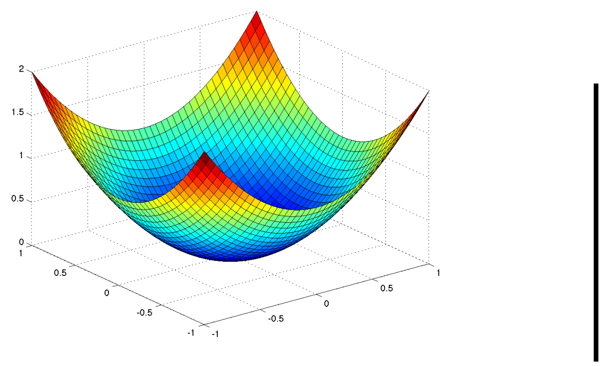
The **Reeb graph** *of the function f, denoted $Reeb(f)$, is the topological space $Graph(f)/\sim$.*

Remark 4. *The Reeb graph of a function f is the set of connected components of level sets of f (with the additional information of the function value of each level set).*

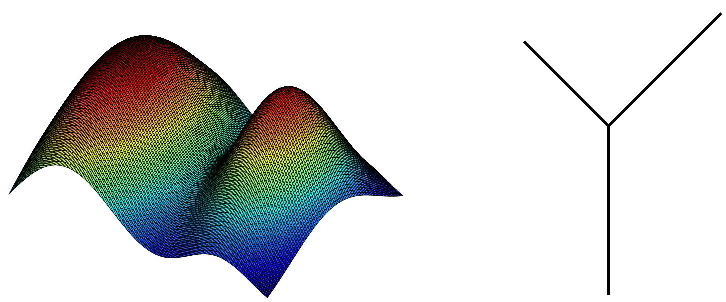
2.2 Examples

We will depict the Reeb graph in the following way: an element $[(x, f(x))] \in \mathrm{Reeb}(f)$ will be represented by a point $p_{[(x,f(x))]}$ in the $x-y$ plane, and if $f(z_1) > f(z_2)$ then the y-coordinate of $p_{[(z_1,f(z_1))]}$ will be larger than $p_{[(z_2,f(z_2))]}$.

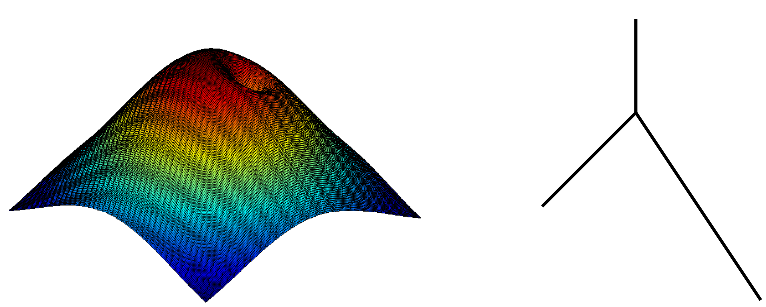
- $f(x_1, x_2) = x_1^2 + x_2^2$



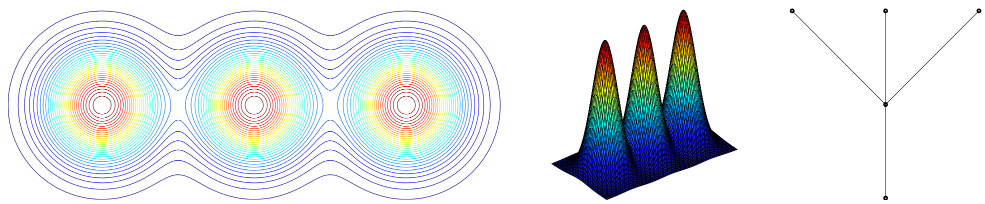
- $f(x_1, x_2) = \exp\left[-(x_1^2 + x_2^2)\right] + \exp\left[-((x_1 - 1)^2 + x_2^2)\right]$



- $f(x_1, x_2) = \exp\left[-(x_1^2 + x_2^2)\right] - 0.1 \exp\left[-10((x_1 - 0.2)^2 + x_2^2)\right]$



- $f(x, y) = \exp\left[-(x_1^2 + x_2^2)\right] + \exp\left[-((x_1 - 3)^2 + x_2^2)\right] + \exp\left[-((x_1 + 3)^2 + x_2^2)\right]$



2.3 Properties of Reeb Graphs

Lemma 1 (Reeb graph is connected)**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then Reeb(f) is connected.*

Lemma 2 (Reeb Tree)**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then Reeb(f) does not contain cycles.*

Remark 5. *Both of these results follow from basic results in topology, namely, that connectedness and contractibility of loops are preserved under quotienting. That is, since Graph(f) is connected and loops in Graph(f) are contractible (so long as f is continuous), we have that Reeb(f) = Graph$(f)/\sim$ must also have these properties.*

Assume now that $f : \mathbb{R}^2 \to \mathbb{R}$ is a Morse function whose critical points have **distinct values**, then we may associate an *attributed graph* to Reeb(f).

Definition 8 (Attributed Graph)**.** *Let $G = (V, E)$ be a graph (V is the vertex set and E is the edge set), and L be a set (called the* label set*). Let $a : V \to L$ be a function (called the* attribute *function). We define the* attributed graph *as $AG = (V, E, L, a)$.*

Definition 9 (Attributed Reeb Tree of a Function)**.** *Let V be the set of critical points of f. Define E to be*

$$\begin{aligned} E = \{(v_i, v_j) : i \neq j, \exists \text{ a continuous map } \gamma : [0,1] \to \text{Reeb}(f) \text{ such that} \\ \gamma(0) = [(v_i, f(v_i))], \gamma(1) = [(v_j, f(v_j))] \text{ and } \gamma(t) \neq [(v, f(v))] \text{ for all } v \in V \text{ and all } t \in (0,1)\}. \end{aligned} \quad (1)$$

Let $L = \mathbb{R}^+$, and

$$a(v) = f(v).$$

Definition 10 (Degree of a Vertex)**.** *Let $G = (V, E)$ be a graph, and $v \in V$, then the degree of a vertex, $deg(v)$, is the number of edges that contain v.*

Theorem 3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^+$ be a Morse function with distinct critical values. Let $(V, E, \mathbb{R}, f)$ be its Attributed Reeb Tree. Then*

1. *(V, E) is a connected tree*
2. *$n_0 - n_1 + n_2 = 2$ where n_0 is the number of maxima, n_1 the number of saddles and n_2 the number of minima*
3. *If $v \in V$ and v is a local minimum/maximum, then $deg(v) = 1$*
4. *If $v \in V$ and v is a saddle, then $deg(v) = 3$*

Remark 6. *Property 2 above is a remarkable fact from Morse Theory, which is more general than it is shown above. Indeed, given a compact surface $S \subset \mathbb{R}^3$, the number $n_0 - n_1 + n_2$ is the same for any Morse function $f : S \to \mathbb{R}^+$, i.e., $n_0 - n_1 + n_2$ (although seemingly a property of the function) is an* **invariant** *of the surface S.*

Remark 7. *Using the fact that for any tree (V, E), we have that $|V| - |E| = 1$ and Property 2, we can conclude by simple algebraic manipulation that $deg(v) = 3$ for a saddle.*

Theorem 4 (Stability of Attributed Reeb Tree Under Noise)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Morse function and set $g_\varepsilon = f + \varepsilon h$ where $h : \mathbb{R}^2 \to \mathbb{R}$ is C^2. Then for all ε sufficiently small, g_ϵ is Morse and $ART(f) = ART(g_\epsilon)$.*

2.4 Diffeomorphisms and the Attributed Reeb Tree

Definition 11 (Diffeomorphism of the Plane). *A function $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a* **diffeomorphism** *provided that $\nabla\psi(x)$ and $\nabla\psi^{-1}(x)$ exists for all $x \in \mathbb{R}^2$.*

Theorem 5 (Invariance of ART Under Diffeomorphisms). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Morse function and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. Then $ART(f) = ART(f \circ \psi)$.*

Remark 8.

- *Note that if p is a critical point of f, then $\psi^{-1}(p)$ is a critical point of $f \circ \psi$:*

$$\nabla(f \circ \psi)(\psi^{-1}(p)) = \nabla\psi(\psi^{-1}(p)) \circ \nabla f(\psi(\psi^{-1}(p))) = \nabla\psi(p) \circ \nabla f(p) = 0 \text{ if } \nabla f(p) = 0.$$

  Therefore the vertex set in the ART of both f and $f \circ \psi$ are equivalent.

- *Moreover, if γ is a continuous path in $f^{-1}(f(x))$ then $\psi \circ \gamma$ is a continuous path in $(f \circ \psi)^{-1}(f \circ \psi(x))$, as diffeomorphisms do not break continuous paths. Therefore, the edge sets in the ART of f and $f \circ \psi$ are equivalent.*

Theorem 6. *If $f, g : \mathbb{R}^2 \to \mathbb{R}$ are Morse functions with distinct critical values and if $ART(f) = ART(g)$, then there exists a monotone function $h : \mathbb{R} \to \mathbb{R}$ and a diffeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f = h \circ g \circ \psi$.*

Remark 9. *By the Morse Lemma, we can construct diffeomorphisms ψ_i around critical points, the idea is then to "stitch" these diffeomorphisms up with "patches" to form the diffeomorphism ψ of interest.*

Theorem 7 (Reconstruction of Function from ART). *If (V, E) is a tree such that each vertex $v \in V$ is of degree 1 or 3, then there exists a Morse function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $ART(f) = (V, E)$.*

Definition 12 (Orbit Space). *Let X be a set, and G be a group.*

- *G* **acts on** *X if each $g \in G$ is also $g : X \to X$ such that*
  1. *For each $g, h \in G$ and $x \in X$, $(gh)x = g(hx)$.*
  2. *For the identity element $e \in G$, we have $ex = x$ for all $x \in X$.*
- *If G acts on X, then the* **orbit** *of a point $x \in X$ is $Gx = \{gx : g \in G\}$.*
- *Define an equivalence relation in X by $x \sim y$ if there exists $g \in G$ such that $gx = y$. The* **orbit space** *(or the quotient of the action G) is the set $X/G = \{[x] : x \in X\}$.*

Theorem 8. *Let $\mathcal{F}$ be the set of Morse functions with distinct critical values, $\mathcal{H}$ denote the set of monotone functions $h : \mathbb{R} \to \mathbb{R}$, and $\mathcal{W}$ denote the set of diffeomorphisms of the plane. Then*

- *$\mathcal{H} \times \mathcal{W}$ acts on $\mathcal{F}$ through the action : $(h, w)f := h \circ f \circ w$ for $h \in \mathcal{H}$, $w \in \mathcal{W}$, and $f \in \mathcal{F}$.*
- *The orbit space $\mathcal{F}/(\mathcal{H} \times \mathcal{W}) = \mathscr{T}$ where $\mathscr{T}$ is the set of trees whose vertices have degree 1 or 3.*

Remark 10. *The second result above is simply a restatement of the Theorems above. Indeed, we can define the mapping $ART : \mathcal{F}/(\mathcal{H} \times W) \to \mathscr{T}$ by*

$$ART([f]) := ART(f), \text{ where } [f] = \{(h, w)f \in \mathcal{F} : (h, w) \in \mathcal{H} \times \mathcal{W}\}$$

The function above is well-defined since by Theorem 5, any representative $g \in [f]$ will have the same Attributed Reeb Tree. Note

- *Theorem 6 states that $ART : \mathcal{F}/(\mathcal{H} \times W) \to \mathscr{T}$ is injective.*
- *Theorem 7 states that $ART : \mathcal{F}/(\mathcal{H} \times W) \to \mathscr{T}$ is surjective.*
- *Therefore, $ART : \mathcal{F}/(\mathcal{H} \times W) \to \mathscr{T}$ is a bijection and therefore, $\mathcal{F}/(\mathcal{H} \times W) = \mathscr{T}$.*